Metaphysics and Method in Dimensional Analysis, 1914-1917

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Abstract

Philosophers of science have largely ignored a foundational and powerful method for physics, dimensional analysis. This paper investigates the methodological and metaphysical foundations of dimensional analysis as they came to salience in a debate last century. In particular, the failed attempt of Richard Tolman to install the principle of similitude—the relativity of size—as the founding principle of dimensional analysis both clarifies the method (and limits) of dimensional analysis and articulates two metaphysical positions regarding quantity dimensions. One position is quantity dimension fundamentalism. This combines a substantival realism with a commitment to a construction principle: there is a set of objectively fundamental quantity dimensions which provide a basis for the construction of derived quantity dimensions. The opposing position, developed primarily by Bridgman, is quantity dimension conventionalism. Bridgman"s conventionalism combines an anti-realism regarding quantity dimensions with a denial of an objectively determined set of basic quantity dimensions. These metaphysical issues were left somewhat unsettled. It is shown here that both of these positions face serious problems: fundamentalism faces epistemological issues regarding our knowledge of the basic quantity dimensions, failing to be properly connected to scientific practice; conventionalism fails to take seriously the empirical constraints on chosen dimensional systems and fails to make dimensional analysis *explanatory*. In their place I put forward an alternative position which saves what is right in both: quantity dimension functionalism. This functionalism allots quantity dimensions a structural, nomological reality and is found to cohere well with their formal structure, allowing for a synthesis of two methodological conceptions of dimensional analysis that *prima facie* are in tension: that dimensional analysis is a *logical* method and that dimensional analysis provides *explanations*.

Contents

1 Introduction

This paper studies a dispute about the methodological foundations of dimensional analysis in order to clarify its *metaphysical* foundations. In particular, consideration of the debate started by the failed attempt of Richard Tolman to install the principle of similitude—the relativity of size—as the founding principle of dimensional analysis both clarifies the method (and limits) of dimensional analysis and articulates two metaphysical positions regarding the status of quantity dimensions. One view, which I call fundamentalism, holds that the basic quantity dimensions are metaphysically robust and fundamental natural kinds. Another view, conventionalism, holds that the basic quantity dimensions are decided by convention and that a system of dimensions has no metaphysical significance but serves as a guide to merely formal unit conversions. Objections to both positions presented in the historical debate are found to have (limited) validity and a third, alternative position, functionalism, is introduced. Quantity dimension functionalism is found to cohere well with the formal structure of quantity dimensions and allows for a synthesis of two methodological conceptions of dimensional analysis that *prima facie* are in tension: that dimensional analysis is a *logical* method and that dimensional analysis provides *explanations*.

The historical discussion will be restricted to the debate prior to Bridgman's landmark *Dimensional Analysis* and will focus largely on an exchange between Bridgman and Tolman.[1] Other significant contributors to the debate, Edgar Buckingham and Tatiana Ehrenfest-Afanassjewa, cannot be given their full due. Connections to general, foundational, and contemporary issues in the metaphysical and mathematical foundations of dimensional analysis will be spun out of this narrative.

In what remains of this introduction, I will introduce dimensional analysis as a method for problem solving in physics, clarify its role as a logical method, and clarify an all important and not often made distinction between unit systems and dimensional systems. This introduction provides all the necessary background for the rest of the paper to follow.

1.1 Dimensional Analysis in Action

Dimensional analysis is well known to even beginning students in physics, though explicit instruction in the method is far from universal. Dimensional analysis finds use in (often heuristic) arguments in fundamental physics and in technical engineering applications alike. Let's consider an example of (standard) dimensional analysis in action.

Say we are tasked with deriving the equation for the period of oscillation, t, of an arbitrary pendulum. We assume that the system can be adequately described in terms of the following quantities: the mass of the pendulum, m, the length of the pendulum, l, and the constant acceleration

[1] In this way it differs from the brief but more comprehensive account of the debates regarding dimensional analysis in Walter (1990). Her account is more comprehensive in that it covers the debates before and after *Dimensional Analysis*, but it is more myopic in its focus on Bridgman—Rightly so, as Walter's book is a biography of Bridgman.

of gravity, g.[2] Next we assume that these quantities are all reducible to mechanical dimensions such that:

$$
\begin{aligned}
[t] &= \mathrm{T} \\
[m] &= \mathrm{M} \\
[l] &= \mathrm{L} \\
[g] &= \mathrm{LT}^{-2}.
\end{aligned}
$$

The square brackets are a function from quantities to their dimensions, here given in terms of the basic mechanical dimensions, mass, length, and time (capital un-italicized letters denote dimensions).[3]

We can restate the problem as that of finding the form the of the function f such that $t = f(m, l, g)$, and so . We assume that this function f takes the form of a monomial $km^{\alpha}l^{\beta}g^{\gamma}$, with numerical scale factor k.[4] With these assumptions, the orthodox founding principle of dimensional analysis requires that $[t] = f([m], [l], [g])$. This is the principle of dimensional homogeneity:

> (The Principle of Dimensional Homogeneity) Every representationally adequate physical equation is dimensionally homogeneous, and an equation is dimensional homogeneous iff the quantity terms[5] on each side have the same dimension.[6]

The principle of dimensional homogeneity therefore defines a set of linear equations to be solved for the *exponents* that t has each of the indicated basic quantity dimensions,

$$
\begin{aligned}
\mathrm{M} &: 1\alpha + 0\beta + 0\gamma = 0 \\
\mathrm{L} &: 0\alpha + 1\beta + 1\gamma = 0 \\
\mathrm{T} &: 0\alpha + 0\beta - 2\gamma = 1,
\end{aligned}
$$

where the Greek variables stand for the exponents of the variables in the monomial and their coefficients are the exponent of the indicated basic quantity dimension had by the corresponding

[2] This condition of "adequate description" is often called "completeness" (e.g. Buckingham 1914). That phrasing gives the wrong idea. Dimension analysis requires only that all of the *relevant* quantities are considered, many quantities that are also descriptive of the system (indeed there is an infinity of them) are excluded due to irrelevance or redundancy, etc. In this way dimensional analysis is a modeling practice (see Pexton 2014).

[3] There is a slightly different convention, following Maxwell (2002), in which $[L]$ represents the length dimension rather than L, etc.

[4] This is due to Bridgman's (1931) lemma, see Berberan-Santos and Pogliani (1999) and Jalloh (Forthcoming) for discussion.

[5] Each of these terms are monomials of quantity variables (or constants) and dimenisionless scale factors, addition and subtraction distinguish terms. This captures the intuition that it makes no sense to add a length to a mass or to subtract a force from a velocity, etc.

[6] This principle is first made explicit by Fourier in his *Théorie Analytique de la Chaleur*: "It must now be remarked that every undetermined magnitude or constant has one dimension proper to itself, and that the terms of one and the same equation could not be compared, if they had not the same exponent of dimension." (Fourier 1878, 128) For more on the geometrical roots of dimensional analysis see De Clark (2017) and Roche (1998).

quantities m, l, and g. By inspection $\alpha = 0$. Now with two equations and two variables (β and γ) we find the solution to be $\beta = 1/2$ and $\gamma = -1/2$, so

$$t = k\sqrt{\frac{l}{g}}$$

where k is some undetermined dimensionless constant. QED.[7]

1.2 Dimensional Analysis as Logic

Dimensional analysis was commonly thought of as a *logical* method by those who developed its foundations (see also Gibbings 1982). I've attempted, in the demonstration above, to make the logical character of dimensional analysis evident by distinguishing assumptions which draw upon our prior physical knowledge and the workings of dimensional analysis itself. In discussing his foundational paper on dimensional analysis (Buckingham 1914), Buckingham wrote:

> Some three or four years ago, having occasion to occupy myself with practical hydro- and aerodynamics, I at once found that I needed to know more about the method in order to use it with confidence for my own purposes...
>
> I had therefore, as it were, to write an elementary textbook on the subject for my own education. My object has been to reduce the method to a mere algebraic routine of general applicability, making it clear that Physics came in only at the start in deciding what variables should be considered, and that the rest was a necessary consequence of the physical knowledge used at the beginning; thus distinguishing sharply between what was *assumed*, either hypothetically or from observation, and what was mere logic and therefore certain. (Buckingham to Rayleigh, November 15 1915)[8]

It is clear from this that Buckingham understood dimensional analysis as a logical method insofar as it was certain and so did not depend on any further empirical claims, i.e. *a priori*. Modeling dimensional analysis on deductive logic, we can say that it provides a form of valid argument (more abstractly, transformation rules): *if* such-and-such quantities have such and such dimensions, relative to a dimensional system (see next section), *then* they are related by so-and-so functions.[9] In

[7]Such derivations can be done more systematically by way of the Π-theorem, a fundamental result of dimensional analysis, which informs us that for any system the number of quantities that describe the system, N, and number of basic dimensions from which the dimensions of those quantities are derived, B, determine the number of dimensionless Π-terms (Π for products of powers of the N quantities) which are sufficient to describe the system: $N - B$. Given that there are four variables and three basic quantity dimensions (M, L, and T), one such dimensionless Π term is needed to describe this pendulum example, the ratio of the period to the function f (i.e. $\sqrt{\frac{l}{g}}$). The Π-theorem is discussed in more somewhat more detail and references are provided in §2.2.

[8]Courtesy of the American Institute of Physics, Niels Bohr Library and Archives, MP 2017-2296; 33.

[9]That the generation of Π-terms and so functional relations can be computed completely and without arbitrariness is shown in Gibbings (2011). That does not mean, of course, that in ordinary practice there is not an art in determining *which* Π-terms and so functional relations are of interest for the relevant system.

our extended post-logical-empiricism hangover, such a distinction may seem hopeless, and worse, old-fashioned—we cannot accept Buckingham's conception of dimensional analysis.[10]

Here I'd like to rehabilitate an idea of dimensional analysis as logic, by abandoning Buckingham's epistemic conception of logic, while accepting that it stands apart from ordinary physics in an important way. The relations between dimensional analysis and experiment are too complex to segregate dimensional analysis from empirical assumptions, but there is still a sense in which dimensional analysis stands above (or below) the ordinary practice of physics in a way similar to relative standing of logic and ordinary reasoning. For this rehabilitation, I will draw on Gil Sagi's (2021) recent defense of an exceptionalist conception of logic *as a methodological discipline*— this contrasts from the usual exceptionalist conceptions of logic on an epistemic basis, like that it is *a priori*, that is now so unfashionable after Quine (1951). In adding dimensional analysis to the roster of methodological disciplines, I am accepting the invitation left open by Sagi that "[p]erhaps there are other methodological disciplines targeting scientific practice" (2021, 9741). I offer the claim that dimensional analysis is the methodological science peculiar to quantitative science, here narrowly considered as peculiar to quantitative *physical* science, and so can synonymously be understood as the *logic of quantities*.

What is a methodological discipline? We may do well to start with the characterization given by Sagi:

> As a start, by a methodological discipline, I mean a discipline that produces tools, methods or a methodology for some practice. I take a method to be a systematic procedure or system of rules for carrying out a practice. There may be methods for very specific practices (measuring the distance between the earth and the moon, solving differential equations) or general methods advising a whole discipline (how to conduct a scientific experiment, how to prove a mathematical theorem)... A methodology, in general, is aimed at a higher level of scientific practice, as it concerns the production and selection of scientific theories. A methodology, I assume, may give rise to a method (for, e.g., theory choice) or consist of a compendium of methods (for reasoning in science). (Sagi 2021, 9736)

A methodological discipline is defined *relationally* to what we may call a *client* discipline. The methodological discipline aids practitioners in aligning their scientific practice to the aims of their first-order client discipline. Put differently, the aims of a methodological discipline are to ensure that the products of some client discipline (e.g. theories or models) meet the internal aims of that client

[10] In a later letter to Rayleigh on January 7 1916, Buckingham already expresses his feeling that his methodological strictures chafed against the zeitgeist: "It is evidently desirable that this subject should receive a clear exposition. Tolman does not, I imagine, care much for the distinctions between known facts, assumptions made for the sake of building up theories, and purely logical operations on these facts or assumptions. And it seems that many of the very clever rising generation of physicists have much the same feeling. I, on the other hand, regard these distinctions as very essential to clear thinking and sound progress." (p 6)

discipline (e.g. prediction, explanation). Here I am proposing that dimensional analysis has physics as a client discipline (among others)—dimensional analysis provides principles and derivational techniques that allow physicists to check the validity of their quantitative equations and to efficiently derive new ones.[11]

What is the relation between a methodological discipline and a client discipline? One intriguing characterization of the relation between the two that Sagi gives involves an extension of the use-mention distinction: client disciplines *use* tools, methods, and concepts that are *mentioned* (e.g. criticized, constructed) by the corresponding methodological discipline. While physics uses concepts of quantity, principles of homogeneity, and dimensional systems, it is left for dimensional analysis to discuss the nature of quantities, justify and determine the consequences of dimensional homogeneity (e.g. the Π-theorem), and elaborate and distinguish dimensional systems.[12] It is important that this exceptionalist, relational conception of methodological disciplines does not lapse into a sort of epistemic foundationalism as attacked by Quine. We can capture both the special position of a methodological discipline and its revisability by distinguishing two phases of research:

> (Business as Usual) The methodological discipline constructs, describes, and regiments the techniques and concepts used by the client discipline. The rules set by the methodological discipline exert normative force on the practitioners of the client discipline, when there is a discrepancy, the principles set by the methodological principle take precedence.
>
> (Negotiation) First order problems or developments in the client discipline lead to a reconsideration of the principles of the methodological discipline and the relationship between the two—neither discipline takes normative priority to the other.

In the Business as Usual phase the client-provider relation is as expected, the methodological discipline provides tools and method which hold normative force over the practices of the client discipline—a equation of physics found to violate dimensional homogeneity is an equation to be corrected (or at least used with great care in special circumstances). In the Negotiation phase, usual business is disrupted, internal pressures from the client discipline (e.g. empirical results, paradoxes) lead to adjustments in the methodological principle and even shifts in what aspects of the relevant scientific practice belong to which discipline. The historical episode to be considered here is usefully described in these terms: In the early twentieth century, pragmatic matters (above all the development of airplanes) lead to a formalized business deal between the nascent methodological discipline of dimensional analysis and the physical sciences. While this deal quickly came to be "business as usual", Tolman attempted in 1914 to renegotiate the deal. Inspired by radical

[11] A similar distinction between "framed" and "framing" inquiry has been articulated and defended by Henne (2023).

[12] A closely related and analogous methodological discipline is metrology, which provides the (experimental) physicist with units of measurement, values for constants, rules for error propagation, etc. Metrology is an important case to consider as the divide between the methodological discipline and the client disciplines has there become sociologically and institutionally regimented in a clarifying way.

developments in the client discipline, physics, Tolman attempted to augment the foundations of the methodological discipline with a new relativity principle and thereby provide new constraints on the client discipline. While Tolman's negotiation failed, it made explicit many implicit aspects of the initial deal between dimensional analysis and physics, some which have still yet to be fully clarified. In the next subsection I clarify an important aspect of the usual deal and raise one issue left to be negotiated: To what extent are features of our dimensional systems objective?

1.3 Dimensional Systems and Unit Systems

Dimensional analysis depends on some assumptions regarding physical quantities. They must form a complete dimensional system, meaning that the complete set of quantities are reducible to products of powers of fundamental units multiplied by a numerical scale factor:[13]

$$Q_i = k_i u_a^{\alpha} u_b^{\beta} u_c^{\gamma} \dots$$

Q_i is some arbitrary quantity. k_i is some numerical factor. u_x is some fundamental unit. The Greek exponents are known as dimensions, following Fourier (1878).[14] Each basic unit is assigned a basic dimension. For example, in a mechanical dimensional system,

$$m = u_{\mathrm{M}}$$
$$l = u_{\mathrm{L}}$$
$$t = u_{\mathrm{T}}$$

where l, m, and t are arbitrary mass, length, and time quantities, e.g. a kilogram, a meter, and second. Each of these units have a basic dimension,

$$[m] = \mathrm{M}$$
$$[l] = \mathrm{L}$$
$$[t] = \mathrm{T}$$

which, in abstraction from the actual units, we can use to derive the dimension of all other mechanical quantities. Hence dimensional systems, which are determined by the basic dimensions, are more coarse-grained than unit systems. For each dimensional system there is arbitrarily large set of logically possible coherent unit systems which are all inter-convertible and hence form what I will call a "dimensional group".[15] For example, the dimensions of force, F, and the dimensions of velocity,

[13] See Bridgman (1931) and Berberan-Santos and Pogliani (1999) for proofs.

[14] This sometimes leads to expressions like "has exponent d in dimension X" which are equivalent to expressions like "has dimension X^d".

[15]

V, are given so:[16]

$$[F] = \mathrm{MLT}^{-2}$$
$$[V] = \mathrm{LT}^{-1}$$

These dimensional formulae correspond to definitions of mechanical units:

$$f = k_f m l t^{-2}$$
$$v = k_v l t^{-1}.$$

For a *coherent* system of mechanical units $k_f = k_v = 1$.[17] We can distinguish basic quantities, which have dimensional exponent 1 in only one of the basic dimensions, and derived quantities, which have arbitrary dimension in any of the basic dimension. Basic quantities are measured by fundamental units and derived quantities are measured by defined units. The dimensions of the derived quantities encode formal relations between them and the basic quantities. These relations are formal because they identify the transformation relations between derived quantities upon changes in the fundamental units.

For any derived mechanical quantity, Q, its defined unit, q, will be a monomial function of the fundamental units, just as described above:

$$q = m^\alpha l^\beta t^\gamma$$

The Greek dimensional exponents determine how the defined unit changes with arbitrary scalar transformations of the fundamental units:

$$\frac{q'}{q} = \left(\frac{m'}{m}\right)^\alpha \cdot \left(\frac{l'}{l}\right)^\beta \cdot \left(\frac{t'}{t}\right)^\gamma$$

where the primed units are the new units. If we halve the fundamental time unit, $2t' = t$, and leave the mass and length units unchanged, for example, the unit of force, f, will quadruple because

[16] Italicized capital letters are variables for quantities, I will, for the remainder of this section, retain lowercase variables for units. Unitalicized capital letters represent dimensions.

[17] The usage of the terminology "complete" and "coherent" varies widely. I am also here making a distinction between dimensional and unit systems that is not usually made, though see Abraham (1933). I reserve "complete" for dimensional systems with a reduction base as I go on to describe. I reserve "coherent" for any unit system of a complete dimensional system such that the derivative quantities are defined with dimensionless scale factors $k_i = 1$. Complete equations, which are interpreted according to a complete dimensional system, are unit-invariant (in algebraic form) for any coherent unit system of *that* dimensional system. This captures the lessons of Grozier (2020), though he does not make the distinctions I make, as the mistakes he diagnoses could be avoided by the recognition of the distinction between dimensional systems and the more fine-grained unit systems.

$\gamma_f = -2$ and the velocity unit, v, will double because $\gamma_v = -1$:

$$\frac{f'}{f} = \left(\frac{m'}{m}\right)^1 \cdot \left(\frac{l'}{l}\right)^1 \cdot \left(\frac{t'}{t}\right)^{-2} = \left(\frac{2t}{t}\right)^{-2} = \frac{1}{4}$$
$$\frac{v'}{v} = \left(\frac{m'}{m}\right)^0 \cdot \left(\frac{l'}{l}\right)^1 \cdot \left(\frac{t'}{t}\right)^{-1} = \left(\frac{2t}{t}\right)^{-1} = \frac{1}{2}$$

The use and operation of these unit transformation rules and their duality with dimensional formulae are uncontroversial. While much of the methods that dimensional analysis provides to physics are uncontroversial, there remains controversy regarding the *meaning* of its subject matter, quantity dimensions and dimensional formulae.

One interpretation of dimensional analysis harks back to Buckingham's conception of dimensional analysis as a *formal* logic concerned with conventionally decided transformation rules on defined or stipulated "objects". Ultimately dimensional formulae are understood to be formal, rules for the use of units and numerical representations of quantities, which are purely conventional. On this reading, representations of dimensions like M are purely syntactic shorthand for change ratios like m'/m. The basis of a dimensional system and the corresponding formulae for derived dimensions are reducible to rules of translation between ultimately conventional unit systems that regiment our practice of assigning numbers to objects and systems.

There is a competing interpretation of dimensional analysis that holds quantity dimensions to be entities in their own right, irreducible to mere convention and formal rules. On this view dimensional formulae do not only represent unit transformation rules but reveal the metaphysical character of quantities. Not only is a unit of force defined, but a *quantity* of force is *constructed* or *constituted* by the dimensions of mass, length, and time. On this view it is as if the basic dimensions are the fundamental substances from which the more complex derivative quantity dimensions are composed.[18]

In order to further explicate and critically examine these two interpretations of dimensional analytic methods and objects, I will set them against questions regarding the objectivity of the two main features of dimensional systems discussed here: basic quantity dimensions and dimensional formulae.

1.4 Metaphysical Questions

In the debate to be considered a number of metaphysical questions get raised and several metaphysical positions are articulated in response to them. First there is an ontological question, raised in the last section: Are quantity dimensions metaphysically real? I wish to retain this question in this gross form as to not get prematurely distracted by question in the metaphysics of properties. Dimensional

[18]This controversy continues to today, with Skow (2017) arguing against the interpretation of dimensional formulae as denoting constitution relations.

realists answer yes: there is some sort of ontic property that corresponds to e.g. mass that differs from that of e.g. volume. Dimensional anti-realists answer no: quantity dimensions are exhausted by the formal rules given by dimensional analysis for the transformations of units, i.e. dimensional formulae represent unit-change ratios and nothing more. Both the realist and anti-realist positions can be further speciated by consideration of another metaphysical question: Is there a set of objectively basic quantity dimensions (for a given dimensional system)? There is lurking complexity in this question as well; we can distinguish the question of whether there are some particular quantity dimensions that are objectively basic and whether there is some weaker restriction, like cardinality, on the set of basic dimensions. These subquestions will be addressed in what follows, but for coming to the set of metaphysical positions to be consider we need only consider the gross question. The dimensional realist who answers yes, there is a set of objectively basic quantity dimensions, is the quantity dimension fundamentalist, a position articulated in this debate by Tolman. The dimensional realist who answers no is a functionalist, a position I will articulate and defend at the end of this paper. On the anti-realist side, there is the operationalist, who accepts an objectively basic set of quantity dimensions *on an epistemic basis*, rather than a metaphysical one. This position will not be considered here, as it lies outside of the historical scope of this essay.[19] A dimenisional anti-realist that rejects an objective dimensional basis is a thoroughgoing conventionalist—Bridgman articulates this position in opposition to Tolman. The relationships between these metaphysical positions and the questions which they answer are summarized in the following flowchart.

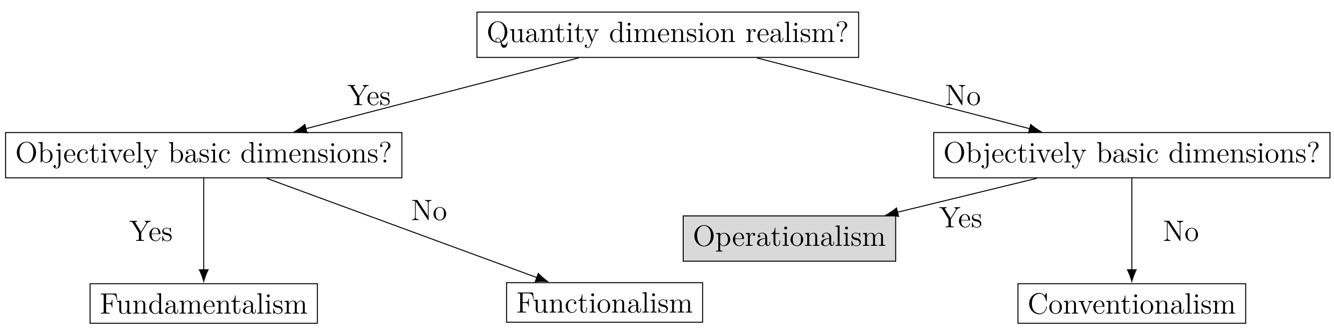

Figure 1: Flow chart through the logical space of quantity dimensions metaphysics.

As I will show, both fundamentalism and conventionalism about quantity dimensions are articulated and defended in the years 1914-1917. A third view, functionalism is presented here as a synthesis of the two, responsive to problems to both historical positions.[20]

[19]The operationalist's basis is determined by the primitive, *direct*, measurement operations that we can perform. Different authors disagree on what dimensions meet this experimental criterion, e.g. fundamental mass vs fundamental force (see, e.g. Gibbings 2011).

[20]Dialectically, one may find this division of the logical space similar to that in Skow (2017). The analogy would be that Skow's positivist stands in for my conventionalist, his contructivist for my fundamentalist, and his definitional connectionist for my functionalist. There are some differences: Skow's definitional connectionist is

Quantity dimension fundamentalism combines a substantival realism[21] with a commitment to a construction principle: there is a set of objectively fundamental quantity dimensions which provide a basis for the construction of derived quantity dimensions.[22]

> (Fundamentalism) The basic quantity dimensions are natural kinds. These kinds are objectively determined independently of the dimensional system we in fact use.

Fundamentalism is just a quantitative counterpart to realism regarding natural kinds, which is common in certain philosophical ecosystems (largely following D. Lewis 1983).[23] These natural kinds are supposed to "cut nature at the joints" in a way independent of human conventions while avoiding the inelegance of grue and similar mongrel properties. This also means that the nature of derived quantity dimensions are completely determined by dimensional formulae, which take on a metaphysical significance beyond their role in guiding unit transformations, e.g. the metaphysical nature of force is described by $[F] = \mathrm{F} = \mathrm{MLT}^{-2}$. For the (putative) basic quantity dimensions, e.g. mass, such formulae are trivial, which points to the basic quantity dimensions having intrinsic essences or quiddities. Whether the derivative quantity dimensions have their own intrinsic quiddities that stand in some sort of grounding relation with the basic dimension quiddities or else are constituted by and "nothing over and above" the basic dimension quiddities depends on the conception of metaphysical construction one adopts, such fine-grained differences are of no concern here.

The major alternative metaphysical position to fundamentalism was most extensively articulated by Percy Bridgman (1916). Bridgman's conventionalism combines an anti-realism regarding quantity dimensions with a denial of there being any objectively determined set of basic quantity dimensions:

> (Conventionalism) Quantity dimensions are purely formal devices that serve as shorthand for unit transformation rules.[24] Being decided by convention, there is no objectively determined set of basic quantity dimensions, nor any objectively determined number of them.

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also a fundamentalist as they are committed to non-relativity, the position that there is an objectively determined basis for our dimensional system. That said, Skow's definitional connectionist comes closer to my functionalist in that he describes quantity dimensions as independent things that are necessarily connected (Skow 2017, 194). An appreciation of the full force of conventionalist symmetries would lead Skow's definitional connectionist to drop the ideas of unique real definitions of derivative dimensions, and so essences of dimensions in general, yielding a functionalist account. A direct confrontation of our arguments will have to be postponed.

[21] I say this to emphasize that this is realism regarding dimensions *qua* substances or natural kinds, as opposed to the functionalist realism I will defend, in which dimensions are something like nomological roles.

[22] I do not probe into different metaphysical accounts of this construction or reduction. I take the relation to at least be as strong as supervenience and further is completely described by the vector space representation of quantity dimensions, see §3.3.

[23] Indeed if, as under most versions of natural kinds metaphysics, the natural kinds supposed are those necessary to establish physics and if physics is essentially quantitative, property fundamentalism just is quantity dimension fundamentalism (e.g. Sider 2011, who generalizes "natural kinds" to "structure").

[24] I.e. change ratios, see Abraham (1933) Grozier (2020).

On this view, unintuitive (or even "unnatural") quantities like jerk (the derivative of acceleration) may serve as a basic dimension just as well as length or time might. Further, the *number* of basic dimensions is a matter of convention, with the elimination or addition of basic quantity dimensions being compensated by a respective elimination or addition of fundamental dimensional constants (see §3.1).

As will be shown, both of these positions face serious problems: fundamentalism faces epistemological issues regarding our knowledge of the fundamental quantity dimensions, failing to be properly connected to scientific practice; conventionalism fails to take seriously the empirical constraints on chosen dimensional systems and fails to make dimensional analysis *explanatory*. This paper closes with an exhibition of a third position which accommodates the surviving aspects of each position:

> (Functionalism) Quantity dimensions are not natural kinds, with essential, intrinsic, or substantival natures but are rather are structural. A quantity dimension is a quantity role in the the laws. A quantity dimension is what is invariant under all of the conventionalist transformations—the ratios of dimensional exponents between quantity dimensions.[25]

By retaining some metaphysical heft, nomological rather than ontological, the functionalist allows for dimensional analysis to be genuinely explanatory—dimensional equations describe the necessary structural relations between properties of physical systems. For the conventionalist, the similarity of distinct systems captured by dimensional analysis is merely a quirk of our measurement system and cannot be attributed to the natures of the systems but merely the unit invariant form of the laws that we have *chosen* to adopt. But the functionalist is also able to capture the conventionalist intuition regarding the absence of an objectively determined basis for a dimensional system: Whether force or mass is treated as basic, it will always be the case that scalings of one will imply scalings of other to the same degree. Scalings of length will correspond to second degree (i.e. squared) scalings of area, and so on.

The context in which the question of the metaphysics of dimensions first came to light was in a debate a regarding which of two principles, the principle of dimensional homogeneity or the principle of similitude, is the fundamental principle of dimensional analysis:

> (The Principle of Similitude) *The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe.* (Tolman 1914a, 244, his emphasis)[26]

[25] By having the exponential ratios be the essence of quantity dimensions, the functionalist nicely coheres with the first formal specification of dimension by Fourier (1878). However, the functionalist holds that these power relations are representations of some nomological relations and is not limited to our mathematical representations of physical systems.

[26] A major warning is to be heeded here. In this paper "the principle of similitude" or "the method of similitude" refers to uses of *Tolman's* principle. More generally "similarity methods" are just another term for using traditional dimensional analysis based on the principle of dimensional homogeneity and proportionality principles (see Sterrett

Tolman's principle of similitude was inspired by the relativity theory (see next section). Tolman conceptualized his principle of similitude as a relativity principle, the relativity of size (or length scale). Tolman assumes, as Poincaré did before him, that a universal scale transformation of lengths ought to be an empirical symmetry, e.g. a doubling of all the lengths overnight would not be empirically detectable. Operationalizing this transformation as a scaling of the length *unit*, Tolman relies on the light postulate (spatializing time) and Coulomb's law (linking the transformation to properties of matter) to spell out the corresponding transformations required in other quantity dimensions to preserve the symmetry (details appear in the next section). As it turns out, Tolman's principle of similitude is false, owing to its conflict with the Newtonian Gravity and the relevant confirming evidence thereof—This was pointed out almost immediately by Buckingham (1914) and amplified by Ehrenfest-Afanassjewa (1916b) and Bridgman (1916). Tolman himself thought a new theory of gravity was imminent.[27] The falsity of Tolman's principle is irrelevant to my concern here, which is the way the methodological debate raised the questions about the metaphysical foundations of dimensional analysis discussed above. So the central question of the debate from my perspective is this: Modulo falsity, does the principle of similitude have any claim to being methodologically prior to the principle of dimensional homogeneity? As this gets cashed out in the debate: Is there a class of problems in which the principle of similitude can be applied and the principle of dimensional homogeneity fails to apply or applies but provides less informative solutions?

2 A Debate in Three Parts

In this section I discuss the debate surrounding Tolman's principle of similitude in three parts, roughly in historical order. Each subsection deals with a dialogue between Tolman and an interlocutor: Edgar Buckingham, Tatiana Ehrenfest-Afanassjewa, and Percy Bridgman. Each dialogue brings forward the metaphysical issues latent in the methodological debate, but special attention is paid to the dialogue with Bridgman, which leads to explicit metaphysical accounts of quantity dimensions.

First a brief note on the scientific context for this debate is necessary. The concern with the foundations of dimensional analysis is connected to other radical changes in the foundations of physics in general.

2017). At the risk of misunderstanding, I am sticking with the terminology used by those in the debate—though it is relatively clear that Buckingham (1914) intended to reclaim the terminology of similitude from Tolman. In the end Buckingham won out.

[27] The relationship between Tolman's principle and the emergence of novel theories of gravity, let alone questions about the nomological nature of the constants (see §2.3), is much too large a topic to be dealt with here in any way. I will only note that Nordström (1915) developed a version of his scalar gravitational theory (an early competitor to GR) that is consistent with Tolman's principle. The development and significance of such a theory is left for future work.

2.1 Contextualizing Dimensional Analysis in the Wake of Relativity

This debate regarding the foundations of dimensional analysis was not about relativity, nor quantum mechanics.[28] That said, it is important for understanding the opening of this line of inquiry to understand some of the fundamental questions that were raised by relativity, which caused Tolman in particular to reconsider the very nature of physical quantities. Walter situates the development of dimensional analysis as part of a broader reckoning with the radical consequences of relativity theory:

> [T]he dimensional analysis controversy revealed a generous amount of confusion about the meaning of relativity and measurement... Einstein's abrogation of the traditional meaning of measurement has demonstrated that the relationship between mathematics and physical reality had to be reconsidered. The dispute over dimensions was just one manifestation of a general concern that would be stated with more precision and politicized by the logical positivists. (Walter 1990, 84)[29]

The following description of this broader context is based on Walter's more thorough accounting of the relevant foundational debates in the wake of relativity.

The special theory of relativity was met with great suspicion and disbelief when it was brought to the attention of American physicists—the promulgation and acceptance of the theory in America is due in no small part to the efforts of Gilbert N. Lewis and Richard C. Tolman in 1908.[30] In Lewis and Tolman's (1909) article, in American pragmatist fashion, describe the principle of relativity as grounded in the generalization of experimental facts—most importantly the Michelson-Morley experiment—and as a principle about what is *measurable*:

> [Einstein] states as a law of nature that absolute uniform translatory motion can be neither measured nor detected. (G. N. Lewis and Tolman 1909, 712)

This is to say that only relative notion has "physical significance". This principle, combined with the postulate of the frame invariance of the speed of light, leads to shocking consequences of relativity

[28] While beyond the scope of this work, see Semay and Willemyns (2021) for an initial look at the application of dimensional analysis to quantum mechanics. While Nordström (1915) moves the debate into one concerning a relativistic theory of gravity, this is not the primary concern of the dimensional analysts. See Porta Mana (2021) for a contemporary and systematic application of dimensional analysis to general relativity theory.

[29] One of the broader trends—interrelated with the dimensions debate—is the search for a natural or ultimate and rationally determinable set of fundamental units. This I cannot discuss here, interested readers should consult Walter (1990).

[30] They presented a paper "Non-Newtonian Mechanics and the Principle of Relativity" at the Christmas meeting of the American Physical Society in 1908, as stated by Kevles (1995, 90). However, I can find no trace of an article in *Physical Review* as he claims. The article (draft completed in May 1909) was published both in *Philosophical Magazine* and *The Proceedings of the American Academy of Arts and Sciences* the following year with an inverted title: "The Principle of Relativity, and Non–Newtonian Mechanics". Here I cite the latter, American publication, a citation for the former can be found in Walter (1990). See also Goldberg (1984) and Goldberg (1987) on the American response to relativity and Lewis' and Tolman's roles.

theory, time dilation and length contraction. Lewis and Tolman's grounding of relativity and its consequences in measurement results leads them to an antirealist interpretation of such consequences:

> Let us emphasize once more, that these changes in the units of time and length, as well as the changes in the units of mass, force, and energy which we are about to discuss, possess in a certain sense a purely factitious significance; although, as we shall show, this is equally true of other universally accepted physical conceptions. We are only justified of speaking of a body in motion when we have in mind some definite though arbitrarily chosen point as a point of rest. The distortion of a moving body is not a physical change in the body itself, but is a scientific fiction. (G. N. Lewis and Tolman 1909, 717)[31]

The contrast drawn is between what they take to be the Einsteinian point of view on these distortion effects and the real contraction of Lorentz.[32] They describe these phenomena as changes in *units* and "in a certain sense psychological". Lewis and Tolman claim that the acceptance of these distortions is the cost of retaining our fundamental conceptions of physics. The psychological unreality of these distortions owes to the fact that their occurrence appears to depend on whether or not some observer considers herself at rest, a judgment lacking in objectivity due to the relativity principle.

The more proper evaluation of the situation is given in Lewis and Tolman's claim that absolute motion has no significance—dilation and contraction are artifacts of an arbitrarily chosen rest point, therby retaining something of our "fundamental conceptions". This is a common feature of symmetry arguments, which occurs in Tolman's argument for the principle of similitude as well as recent debates on quantity symmetries:[33] In arguing for the existence of a symmetry transformation and thereby the unreality of the supposed features of reality that vary under that symmetry, the basis for the symmetry argument seems to be undermined as there is no such feature to be transformed. In Einstein's case this is absolute velocities; In Tolman's case, with his supposed relativity principle, the principle of similitude, it is absolute lengths. This is of course only a matter of charitable interpretation and convenience in discourse: in either case any appearance of self-undermining conclusions can be removing by restating these relativity principles as statements about what objective structure there is. The theory of special relativity rejects any objective, frame-independent, velocity structure. Tolman's principle of similitude rejects any objective, absolute length *magnitudes*, which become dependent on a choice of comparative standard, analogous to how length quantity *values* (i.e. numbers) are relative to a choice of unit standard (i.e. a length defined to be represented by 1).

[31] The special theory of relativity was seen as upending our fundamental concepts of physical *quantities*—when Lewis and Tolman refer to "units" they are conflating the functions of units as reference quantities and as numerical fixed points. The terminology of units vs quantities vs magnitudes was not to be standardized for decades.

[32] As well as Fitzgerald. Recent discussion stemming from Bell (1976) shows that the nature of length contraction and time dilation, or at least their proper ground, is still in question. See e.g. Brown and Pooley (1999).

[33] See Wolff (2020) and citations therein on the absolutism-comparativism debate in the metaphysics of quantity. The supposed mass doubling symmetry at the center of the debate is a direct analogue of Tolman's miniature universe transformation.

2.2 Tolman v. Buckingham

The inciting event is Tolman's (1914a) publication of "The Principle of Similitude" which puts forward a relativity principle—the relativity of size—as the founding principle of dimensional analysis.

> (Relativity of Size) A global transformation of the length scale is both a dynamic and empirical symmetry—there is no objectively determined length scale.

Tolman proceeds by way of a thought experiment: consider two observers O and O' whose measurement standards, ergo their unit systems, stand in such a way that O' assigns the same numerical values to the counterpart quantities in a miniature universe as O assigns to quantities in their universe. Their length measurements, in the unit system of O, will have the relation $l' = xl$. From this and *the acceptance of the speed of light postulate*, their temporal measurements must also stand in the same relation: $t' = xt$. From assuming the invariance of other laws (e.g. Coulomb's law), Tolman derives a whole set of symmetry transformations:[34]

Quantity Kind	**Symmetry Transformation**
Length	$l' = xl$
Time Duration	$t' = xt$
Velocity	$v' = v$
Acceleration	$a' = x^{-1}a$
Mass	$m' = x^{-1}m$
Force	$f' = x^{-2}f$
Energy	$U' = x^{-1}U$
Energy Density	$u' = x^{-4}u$
Electrical Charge	$e' = e$
Entropy	$S' = S$
Temperature	$T' = x^{-1}T$

From these results Tolman determined the functional form of several physical equations describing important physical phenomena: ideal gases, blackbody radiation, the electromagnetic field, and the electron (its mass-radius ratio and its radiation law).[35]

In the same year Buckingham's landmark paper "On Physically Similar Systems" presents the most influential proof of the Π-theorem (a foundational theorem of dimensional analysis), and Buckingham argues that Tolman's principle is only a special case of his result. I will not here go through the derivation of the theorem.[36] Buckingham's statement of essential content of the theorem should be quoted:

[34] Table selectively adapted from Tolman (1915), 226. Note the invariant quantities and the corresponding theoretical commitments of Tolman's principle: the constancy of the speed of light, electromagnetic theory, and the laws of thermodynamics.

[35] Further, in another paper, Tolman (1914b) derived the equation for the specific heat of solids.

[36] See Gibbings (1982, 2011), Sterrett (2009, 2017, 2021), and Pobedrya and Georgievskii (2006).

> When absolute units are used, the validity of a complete physical equation is unaffected by changes in the fundamental units. Hence in changing from a system S to a similar system S' it is immaterial to the validity of the equation in question whether we do or do not retain our original fundamental units. If we alter the sizes of the fundamental units $[Q_1] \ldots [Q_k]$ in the same ratios as the kinds of quantity $Q_1 \ldots Q_k$ which they measure, the numerical value of any quantity of one of these kinds will be the same in both systems. And if we do not change the relations of the derived and fundamental units of our absolute system, every derived unit $[P]$ will change in the same ratio as every quantity P of that kind, so that the numerical value of every quantity in the system S will be equal to the numerical value of the corresponding quantity in the similar system S'. (Buckingham 1914, 354)[37]

While Buckingham here follows the Maxwellian fashion of discussing dimensional analysis in terms of invariance of "complete" equations under transformations of the fundamental units, we can understand his claim here as a generalization of Tolman's similarity principle. Given a coherent or absolute unit system, the relations between basic and derived quantities are defined such that arbitrary changes in the magnitudes of the basic quantities, including the fundamental units, *induce* changes in the derivative quantities, and the derived units, such that representationally adequate equations and dimensionally homogeneous equations, interpreted quantitatively or numerically remain true. This is done without stipulating a *particular* invariance with respect to transformations of the *length* quantities. In brief the theorem states thus: All physical equations are dimensional homogeneous and so can be put in the form:

$$A_1 + A_2 + \cdots + A_N = 0,$$

where each A-term is a product of powers of the fundamental Q-terms (the basic quantities of the dimensional system, e.g. masses, lengths, and times) and each term has the same dimension: $[A_i] = [A_j]$. Therefore, subtracting A_N and then dividing through by $-A_N$ yields an equation with dimensionless Π-terms:[38]

$$\Pi_1 + \Pi_2 + \cdots + \Pi_{N-1} = 1.$$

These dimensionless Π-terms will be invariant under any change of *numerical value* (passive trans-

[37]Walter's discussion contains a claim which requires correction. Walter distinguishes similitude, "a simple way to investigate the manner in which a change of scale affects the properties of physical systems", from dimensional homogeneity, which requires that "the operation of addition and the relationship of equality are valid only for objects [i.e. quantities] of the same kind [i.e. dimension]" (Walter 1990, 86–87). The claim to be criticized is that "Buckingham, like everyone else" conflated these two bits of dimensional reasoning. This claim is false: Buckingham (1914) clearly distinguishes similitude and dimensional homogeneity as he uses the principle of dimensional homogeneity to provide a proof of the Π-theorem, which in turn defines a criterion for physical similarity. One follows from the other, but there is no indication that these are to be equated.

[38]The dimensionless quantities and the theorem get their name from the fact that the dimensionless terms of the equation have the form of product-functions: $\Pi = \prod_i^N Q_i^{x_i}$.

formation) or *magnitude* (active transformation) of the fundamental units. Buckingham makes the case that Tolman's selection of speed, charge, and entropy as the invariants of the symmetries of his dimensional system is merely a specific realization of the general theorem, itself a consequence of dimensional homogeneity.[39] Buckingham goes on to show that his method solves the problems that Tolman's principle of similitude purports to, e.g. the ratio of the mass and radius of an electron. He also goes on to derive the essential inconsistency of Tolman's system and the gravitational law.

Tolman (1915) responds to Buckingham and argues that the Principle of Similitude is distinct from and superior to the Principle of Dimensional Homogeneity on grounds of the latter's *inapplicability* to systems with dimensional constants of unknown dimensionality. These are cases in which dimensional analysis necessitates the introduction of *dimensional* constants in order to make dimensionless products of quantities (i.e. Π-terms). Consider Stefan's law, $u = aT^4$. By the lights of the dimensional analyst, *in advance of the establishment of the dimension of* a, the equation could have a different algebraic form, e.g. $u = aT^3$.

In this case, the dimensional analyst is tasked with determining a function that relates the energy density of a blackbody, u, and its absolute temperature, T. Their respective dimensions, $\mathrm{ML}^{-1}\mathrm{T}^{-2}$ and Θ, are incommensurable, so the principle of dimensional homogeneity is of no help. *If* the dimensional analyst was give the dimensions of the constant, a, which mediates relation between the two constant, dimensional homogeneity would then constrain the form of the functional relation between the two. *If* the dimensional analyst was given the form of the functional relationship between u and T, then dimensional homogeneity would constrain the dimensions of the mediating constant, a. Without either the dimensions of the mediating constant or the form of the function relating the two inhomogeneous quantities, the dimensional analyst armed only with the principle of dimensional homogeneity can make no derivations.

On the contrary, the principle of similitude tells us that u must be numerically equivalent to its scale counterpart, u':[40]

$$u = F(T) = u' = F(T') = x^4 F(x^{-1}T).$$

The solution for this equation requires taking temperature to the fourth power, and due to the ratio structure of the quantities involved, the equation is only fixed up to a scalar factor, a,[41] yielding

[39] Further: "The unnecessary introduction of new postulates into physics is of doubtful advantage, and it seems to me decidedly better, from the physicist's standpoint, not to drag in either electrons or relativity when we can get on just as well without them." (Buckingham 1914, 356) Ehrenfest-Afanassjewa (1916b) makes the same criticism. This can be understood as opposition to Tolman's attempt at renegotiating the lines between the dimensional analysis (a methodological discipline) and first order physical laws (which make up the client discipline).

[40] Referring to the table above we see that u scales with x^4 and T with x^{-1}.

[41] One way to think about the nature of the functional results yielded by either form of dimensional analysis is that the results give the family of curves that corresponds to the function, but doesn't give you the value of the coefficients. Those are found by experiment (see Gibbings 1974; Gibbings 2011 on the relation of dimensional analysis to experiment).

Stefan's law:

$$u = aT^4.$$

Now considerations of dimensional analysis non-arbitrarily yield the dimensions of the constant. As the dimensional analyst starts with neither the form of the equation nor the dimension of the constant, *the principle of dimensional homogeneity is not applicable*. If the dimensional analyst had the form of the law, the constraint of dimensional homogeneity would immediately yield the dimensions of the constant. If the dimensional analyst has the dimensions of the constant, the constraint of dimensional homogeneity would (up to a scale factor) determine the functional, algebraic form of the equation, as is usually done in dimensional derivations.

Tolman puts the relation of the two principles thus:

> Where dimensional constants enter, the principle of dimensional homogeneity is of no avail in predicting the form of a relation, since we cannot tell beforehand what the dimensions of the constant are going to be. For such problems we must have recourse to the principle of similitude. On the other hand, when dimensional constants do not enter into the relation, although we may apply either principle, the principle of similitude is usually the less powerful since it merely prescribes invariance when the different measurements are multiplied by powers of a single arbitrary multiplier x, while the principle of dimensional homogeneity prescribes the more drastic requirement of invariance when the multiplications are carried out with a different arbitrary multiplier for each fundamental property. (Tolman 1915, 232)

The principle of dimensional homogeneity requires a wider range of symmetry transformations (it has a wider class of models, i.e. allows for mappings between worlds which would violate the laws under Tolman's principle of similitude) and so is the more powerful principle, but the principle of similitude applies to cases that the principle of dimensional homogeneity does not and so is more apt to be placed as the fundamental principle of dimensional analysis.

2.3 Tolman v. Ehrenfest-Afanassjewa

There is an interpretative issue that will bring us back to the metaphysical considerations at hand. Ehrenfest-Afanassjewa[42] most clearly states an objection to Tolman's principle shared by the other

[42] Walter's (1990) account of this historical debate is overly dismissive of Ehrenfest-Afanassjewa's contributions, especially her later, post-*Dimensional Analysis*, mathematical intervention (Ehrenfest-Afanassjewa 1926), which is only described as "extensive and confusing" (Walter 1990, 101). This dismissal is unfortunately mirrored in responses by Bridgman (1926) and Campbell (1926)—though Bridgman includes Ehrenfest-Afanassjewa (1926) in the list of important references which have appeared in-between editions of *Dimensional Analysis*. (The list can be found in the preface to the revised edition.) A major reconsideration of her work in dimensional analysis is under development, but see also San Juan (1947), Palacios (1964), and Johnson (2018) for developments of her approach to dimensional analysis. See Uffink et al. (2021) for a more general reevaluation of her work in mathematics and physics.

respondents: The principle of similitude is merely a restricted version of the principle of dimensional homogeneity, and surely the more general principle is more fundamental. From the first paragraph:

> An accurate analysis shows that Tolman's considerations possess at least a close connection with the reduction to a definite hypothesis of the conviction of the homogeneity [unit invariance][43] of all the equations of physics, a conviction which is commonly used without any foundation. This is not the intention of the author, as appears from his third paper on the same subject, yet he really does nothing else but construct a system of dimensions of his own (indeed one that in some respects deviates from the C.G.S. system), and he examines all equations with a view to homogeneity *as regards this system of dimensions*. (Ehrenfest-Afanassjewa 1916b, 1)

While Tolman (1916) rejects the presentation of his principle as determining a system of dimensions, he accepts the presentation of the relationship between the two principles: The principle of similitude involves a further empirical *ansatz* which is to be settled by the investigations into the nature of gravity, and the principle is to be given methodological priority due to its usefulness. His parting with Ehrenfest-Afanassjewa provides the opportunity to raise a distinction. When Ehrenfest-Afanassjewa states that Tolman is establishing a principle of homogeneity restricted to a special set of dimensions she is referring to *representational* dimensions—dimensions considered only as change-ratios for class on unit systems. When Tolman claims that this is not the case, he is considering *ontic* dimensions—dimensions considered as descriptions of the nature of the quantities defined in terms of their formulae.

> (Representational Dimensions) Dimensions encode the transformations of numerical representations of quantities due to changes in unit systems. A quantity having a positive dimension of power n in mass means that the numerical value of that quantity will differ in unit systems by the ratio of their mass units to the nth power.

> (Ontic Dimensions) Dimensions are properties of quantities in physical systems; they encode similarity relations between different systems. That two systems share quantities of like dimensions shows that they share some class of symmetry transformation.[44]

We could just as well distinguish these as unit-dimensions and quantity-dimensions.[45] Representational dimensions are merely formal devices translating between unit conventions. As described

[43] Homogeneity, i.e. unit invariance, is sometimes treated as the fundamental principle of dimensional analysis in lieu of dimensional homogeneity. Authors vary on which is to be taken as axiomatic and which is to be derived, but the cases in which unit invariance and dimensional homogeneity come apart are so few and spurious as to be dismissed for our purposes (cf. Bridgman 1931). I treat both approaches as the "dimensional homogeneity" approach. For more on the mathematical definition of homogeneity, see Ehrenfest-Afanassjewa (1926), San Juan (1947), and Palacios (1964).

[44] This distinction is given by Johnson (2018), 105-112. A similar distinction between dimension-first and unit-first attempts to provide a mathematical model for the quantity calculus is noted by Raposo (2018). See also Sterrett (2009) for the connection between similarity relations and ontic quantity dimensions.

[45] This distinction became clearer in the 1930s, see Abraham (1933) and redacted.

before they are exhausted by the exponents in the change ratios of the dimension in question relative to the basic dimensions, e.g. $[F]$ is shorthand for the unit transformation rule $\frac{f'}{f} = \left(\frac{m'}{m}\right)^1 \cdot \left(\frac{l'}{l}\right)^1 \cdot \left(\frac{t'}{t}\right)^{-2}$. Ontic quantity dimensions, according to the fundamentalist at least, are intrinsic properties of physical quantities. Ontic dimensions cut up the world of quantities into natural kinds.

Insofar as Tolman is offering a different formal system upon which to base dimensional analysis, the diagnosis of Ehrenfest-Afanassjewa and others is correct, it is a restricted form of the orthodox system founded on the principle of dimensional homogeneity. The question then is whether the additional content used to restrict the symmetry class is worth having. On this count Tolman's principle fails, it involves too many substantive assumptions—it is committed to the relative *a priori* truth of dynamical principles like Coloumb's law and the light postulate,[46] which is inappropriate for a methodological discipline like dimensional analysis. To be clear, the a priority of these laws is inappropriate to dimensional analysis because they are thought to be empirical laws, i.e. capable of confirmation or disconfirmation, and dimensional analysis is to take on the role of a logic, whose claims are not to be questioned in the ordinary course of physics. As indicated above, the advent of relativity lead Tolman to believe this was *not* a time for physics as usual.

Setting aside for now the dispute over the meaning of "dimension", Ehrenfest-Afanassjewa argues that Tolman's similitude transformations should only be understood as *formal* or *representational*, that is *passive*, transformations, i.e. unit changes.[47] She places conditions on Tolman's active, *ontic* interpretation of these transformations as indicating actual changes in size, e.g. a miniature universe:

(1) that a model universe in the sense defined above is possible,

(2) that we possess all equations which are wanted for a full description of the whole universe,

(3) that the latter condition is especially fulfilled by those equations which in the C.G.S. system serve to fix the dimensions of the different quantities. (Ehrenfest-Afanassjewa 1916b, 4)

To these conditions she raises three objections. First, the unit transformation coefficients (or scale factors) for time, length, and mass (and so on) are fixed independently of any investigation into the possibility of such model universes. Second, the full description condition necessitates that the transformation coefficients (she also says, in quotes, the "dimensions") of the other quantities are fixed by the transformation so that definitions of novel quantities are invariant under such transformations—this unnecessarily reduces the total number of dimensions ("the number of degrees of freedom of the transformation", i.e. the number of independent, basic dimensions). Third, there

[46]In the sense of having increased resistance to empirical disconfirmation—these laws are held closer to the center of the web of belief than otherwise accepted.

[47]"The transition from the numbers x to x'_i may also be thought of in another way: instead of imagining measurements to be made with the same units in two different worlds, we may conceive the measurements to be carried out applying two different sets of units to the same objects ('in the same world')." (Ehrenfest-Afanassjewa 1916b, 3)

is no reason to think that the current fundamental dimensions are sufficient to capture all of nature ("which should give a *necessary* reduction of the degrees of freedom" in the dimensional system), and in the case of Tolman's reduced set of the single dimension of length (for the mechanical quantities), it is insufficient to capture Newtonian gravity.[48]

Tolman accepts Ehrenfest-Afanassjewa's presentation of the issue, with regards to the fact that his principle of similitude ensures invariance under the arbitrary transformation of only one quantity dimension (length) while her (and Buckingham's) principle is more generally arbitrary—the principle of dimensional homogeneity ensures invariance under aribrary transformations of five basic quantity dimensions (under the usual dimensional system incorporating thermodynamics and electromagnetism). As mentioned, Tolman does object to her characterization of his principle as determining another "system of dimensions" distinct from the then standard centimeter-gram-second system.[49] Tolman further expresses a realist, metaphysically robust account of what a system of dimensions is:

> The dimensions of a quantity may be best regarded, I believe, as a shorthand statement of the definition of that kind of quantity in terms of certain fundamental kinds of quantity, and hence also as an expression of the essential physical nature of the quantity in question. If, for example, we define force as mass times acceleration, the dimensions of force will be $[mlt^{-2}]$ and this may be regarded as a shorthand recapitulation of the definition of force in terms of mass, length and time, and also as an expression of the essential physical nature of force. (Tolman 1916)

Tolman argues that the second principle invoked, that the dimensions of a quantity expresses the essential nature of that quantity *grounds* the principle of dimensional homogeneity. That an equation must have terms of equal exponent in each basic dimension on either side follows if equations are taken not only to describe numerical equalities, but also *quantity identities*. Here Tolman assimilates the definition of derived quantity dimensions and their metaphysical constitution. That an assertion of the latter does not unproblematically follow from the latter is discussed in the literature (e.g. Johnson 2018; Skow 2017)—Tolman's conflation of definition and constitution is a target of conventionalist critique.

[48]Ehrenfest-Afanassjewa suggests a strategy for saving the ontic interpretation of the dimensional symmetries—the scaling of dimensional constants so as to guarantee quantity symmetries Jalloh (Forthcoming). The introduced constant can be understood two ways: either as some real quantity, like a postulated constant of matter, or else "denote it as a product of special values of the variables occurring in the equation" (Ehrenfest-Afanassjewa 1916b, 5). I beg off explaining this here, besides indicating that she develops this more thoroughly as the introduction of "formal variables" in Ehrenfest-Afanassjewa (1916a). The upshot: such an extension of the " 'physical' meaning of the constants" trivializes the possibility of active scale transformations and the invariance of equations under such transformations, and so "ceases to afford a criterion for distinguishing between equations which are 'physically allowable' and arbitrary equations"(Ehrenfest-Afanassjewa 1916b, 6). This deserves more exegesis and investigation than I can provide here.

[49]Or rather the dimensional system for which C.G.S. is a coherent units system (see §1.3). In this respect there is no difference between the C.G.S. system and a M.K.S. system.

The metaphysical interpretation of dimensional systems make clear Tolman's reason for denying that the principle of similitude provides one. According to the principle of dimensional homogeneity force is defined and constituted by mass, length, and time, according to the formula: $[f] = \mathrm{MLT}^{-2}$. Under the system of dimensions that would be given by the principle of similitude, force is a function only of length, $[f] = \mathrm{L}^{-2}$. If Tolman were committed to a system of dimensions given by the principle of similitude, he would say the principle attributes force the *nature* of an inverse area. For this reason he later does not say that his principle provides a system of (ontic) dimensions, but rather is an *ansatz* which is useful in some circumstances and whose principal commitment, the possibility of miniature universes, is available for empirical (dis)confirmation, by way of the implied theory of gravity.

2.4 Tolman v. Bridgman

Tolman's *ansatz* is the target of Bridgman's critique:

> If the exact form of the equations and their mode of application should turn out to be exactly identifiable with the corresponding manipulations of the theory of dimensions, then the principle of similitude must be judged not to be new, in spite of the form of statement above. I shall try to show in this note that such an identification is possible; that in so far as the principle of similitude is correct it gives no results not attainable by dimensional reasoning, and that in its universal form as stated above it cannot be correct. (Bridgman 1916, 424)[50]

Bridgman's aim, then, is to show that Tolman's principle of similitude is more widely applicable than the principle of dimensional homogeneity only insofar as it produces *incorrect* results.

Bridgman diagnoses Tolman's apparent examples of the broader applicability of the principle of similitude (Stefan's law, the gas equation, etc.) by drawing attention to a special feature of the dimensional constants involved:

> The principle of similitude may be applied with correct results to all those cases in which the dimensional constants have such a special form that they are not changed in numerical magnitude by the restricted change of units allowed by the principle. (Bridgman 1916, 425)

The dimensions of Stefan's constant, a, are $\mathrm{ML}^{-1}\mathrm{T}^{-2}\Theta^{-4}$, so we can express a as $N_a m l^{-1} t^{-2} \theta^4$, where N_a is some dimensionless number and m, l, t, and θ are units of mass, length, time, and temperature, respectively. Now apply the principle of similitude:

$$a = a' = N_a x m' x l'^{-1} x^2 t'^{-2} x^{-4} \theta'^{-4} = N_a m' l'^{-1} t'^{-2} \theta'^{-4}.$$

[50]Where "the universal form" is the statement that the materials which constitute the universe could be used to create an empirically indistinguishable universe which differed only in size (with respect to the length scale).

The x factors cancel and the numerical value of Stefan's constant is invariably N_a. That only some such constants are invariant under dimensional scale transformations is evident in Tolman's failure to capture Newtonian Gravitation: $G = N_G \mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}$ scales with factor x^{-2}. The conclusion of Bridgman's argument is that the method of similitude requires an assumption regarding the dimensionality of constants just the same as method of dimensional homogeneity does: a user of the principle of similitude must assume that the dimensional constants which figure in the fundamental equations are such that their dimensional transformation coefficients cancel out. This assumption bears out surprisingly often: In addition Bridgman cites the gas constant, the velocity of light, and the constant of quantum action. Is there some metaphysical significance to this seeming conspiracy of the dimensional constants?

Bridgman answers in the negative, the apparent conspiracy can be explained by the dimensional structure of our *defined* unit systems. By limiting valid unit transformations to those that leave that some choice of constants are invariant, e.g. c and e in Tolman's system, a number of consistent systems of dimensions can be defined. Bridgman amplifies Buckingham's observation that the number of independent basic dimensions or units can be determined by the number of unit-invariant quantity relations, i.e. laws, we chose to accept as axiomatic (or *a priori* in the sense indicated above). The number of fundamental quantity dimensions (and so dimensional constants) is to some extent conventional. If force, for example, was to be set as an additional fundamental quantity, we would need to introduce a new dimensional constant to Newton's second law. Instead we take the law, with this would-be constant set to unity, as a unit-invariant axiom.[51] Bridgman argues that we accept dimensional *definitions* not owing to some metaphysical identity but due to the frequency of the corresponding experimental fact.

Bridgman provides a helpful demonstration of the conventionality involved. I will modify his convention of using the square brackets $[x]$ to using curly brackets $\{x\}$ denote the unitless numerical value of x (in line with contemporary standards, see JCGM 2012). Bridgman provides a description of each of the constants of nature in terms of the fundamental units (5 constants and 5 basic units):[52]

$$G = \{G\}m^{-1}l^3t^{-2} = \{G'\}m'^{-1}l'^3t'^{-2}$$
$$c = \{c\}lt^{-1} = \{c'\}l't'^{-1}$$
$$k = \{k\}ml^2t^{-2}\theta^{-1} = \{k'\}m'l'^2t'-2\theta'-1$$
$$h = \{h\}ml^{-2}t^{-1} = \{h'\}m'l'^{-2}t'^{-1}$$
$$E = \{E\}e^{-2}ml^{-3}t^{-2} = \{E'\}e'^{-2}m'l'^{-3}t^{-2}$$

[51] Bridgman and Buckingham point towards Euclid for the origin of this observation. Consider the system in which area is a distinct dimension from length.

[52] G is the gravitational constant; c is the light constant; k is the (Boltzmann) thermodynamic constant; h is the quantum constant; E is the (Coulomb) electric force constant. The following two sets of equations are adapted from Bridgman (1916), 429.

These equations can be used to determine the value of the constants under changes of fundamental units. Or instead they can be reformulated so as to be used in order to determine the unit transformations that keep their values fixed (or changed by whatever ratio we wish to consider):

$$l'^2 = \frac{\{h\}}{\{h'\}}\left(\frac{\{c\}}{\{c'\}}\right)^{-3}\frac{\{G\}}{\{G'\}}l^2$$
$$t'^2 = \frac{\{h\}}{\{h'\}}\left(\frac{\{c\}}{\{c'\}}\right)^{-5}\frac{\{G\}}{\{G'\}}t^2$$
$$m'^2 = \frac{\{h\}}{\{h'\}}\frac{\{c\}}{\{c'\}}\left(\frac{\{G\}}{\{G'\}}\right)^{-1}m^2$$
$$\theta'^2 = \frac{\{h\}}{\{h'\}}\left(\frac{\{c\}}{\{c'\}}\right)^{5}\left(\frac{\{k\}}{\{k'\}}\right)^{-2}\frac{\{G\}}{\{G'\}}\theta^2$$
$$e'^2 = \frac{\{h\}}{\{h'\}}\frac{\{c\}}{\{c'\}}\left(\frac{\{E\}}{\{E'\}}\right)^{-1}e^2$$

Tolman's transformation equations can be derived by holding all constants fixed except for G, which changes by a factor of x^{-2}. However different transformation equations can be defined by varying other constants and holding G fixed. In each of these systems *some* constant or other is the odd man out, i.e. is variant under similitude transformations. Generally speaking, if we wish to freely vary some number of the fundamental units (like Tolman does for length), we will have to vary the same number of universal constants. The indeterminancy of *which* constants are varied due to the conventional choice of *which* fundamental unit to ground our dimensional system in (i.e. a choice of alternative similitude principles) was taken by Bridgman to undermine Tolman's characterization of his principle as an empirical *ansatz* to guide the development of a novel theory of gravity. There is no more reason to hope for a new theory of *gravity* guided by this principle than a new theory of *electricity*. The constant, and so the physical theory, that "the" principle of similitude is in tension with is a matter of arbitrary choice. In other words, the principle does not yield *unique* empirical predictions—which is to be expected given Tolman's retreat to presenting the principle as only defining a formal system of dimensions (see §2.3).

Tolman presents a full-fledged metaphysical account of "measurable quantities" in his final response regarding the principle of similitude. This account is in no way reactionary, but rather is to serve a foundational purpose:

> The time is already ripe for a much more comprehensive and systematic treatment of the field of mathematical physics than has hitherto been attempted, and the completion of this task would make it possible to derive all the equations of mathematical physics from a few consistent and independent postulates, and to define all the quantities occurring in these equations in terms of a small number of indefinables. The purpose of this article is to discuss from a somewhat general point of view the nature of the quantities which occur in the equations of mathematical physics and to consider a set of indefinables for

> their definition. We shall thus hope to help in the preparation for that more complete systematization of mathematical physics which is undoubtedly coming. (Tolman 1917, 237)

Tolman aims to prepare the way for a generally axiomatic treatment of physics as a whole.[53]

Tolman reintroduces his metaphysical posit by way of discussing the relation that holds between fundamental and derived quantities, which is represented by dimensional formulae:

> *The dimensional formula of a quantity may be regarded as a shorthand statement of the definition of that kind of quantity in terms of the kinds of quantity chosen as fundamental, and hence also as a partial statement of the "physical nature" of the quantity in question.* (Tolman 1917, 242, his emphasis)[54]

As basic dimensions have such a metaphysical significance, Tolman holds that the apparent necessity of five fundamental quantity dimensions (three mechanical ones, one for electromagnetism, another for thermodynamics) is due to there being "five fundamentally different kinds of 'thing' ": space, time, matter, electricity, and entropy.

Beyond being sufficient to account for all known physical kinds, Tolman puts forth two further conditions on a chosen set of fundamental quantity dimensions. The fundamental quantities must be extensive—this allows for extensive methods of measurement for all derived quantities even those that are themselves intensive (consider the role of a thermometer in measuring the temperature).[55] The set of fundamental quantity dimensions must also be such that they provide an optimal level of simplicity to the system of quantities.

With all this on the table, Tolman argues that Bridgman's conventionalism is due to a confusion of quantity and unit:

> The fact that it has become usual to pick out the units for derived quantities in the way indicated has sometimes led to an unfortunate confusion as to the real significance of dimensional formulae. Thus there has grown up the practice of speaking of the dimensions of a unit when what is really intended is the dimensions of the quantity involved. It certainly seems best, however, to use the dimensional formula of a quantity as a shorthand restatement of its definition in terms of the fundamental kinds of quantity.

[53] Appropriate to the generality of his aims, Tolman begins by taking on Russell's (1903) distinction of magnitude and quantity. One might say that magnitudes *measure* quantities, i.e. there is a map from quantities to magnitudes that assigns quantities relative locations in magnitude space with sufficient structure to ground the assignment of numbers and an algebra. Generally the literature has not preserved this distinction and instead uses magnitude to refer to the size (numerical or not) of a concrete quantity (but see Tal 2021). Tolman's system, including his fundamental distinction of intensive and extensive quantities cannot be dealt with here.

[54] That dimension can at most only be a partial description of the nature of a quantity is here set aside, see Lodge (1888) and Mari (2009). Tolman later recognizes this, see [redacted].

[55] "In case the derived quantity has intensive rather than extensive magnitude some more or less artificial correlation of the magnitude in question with quantities having extensive magnitude will then have to be used, as has been done in the case of our ordinary temperature scale." (Tolman 1917, 248)

> The dimensional formula is thus a symbol for the physical nature of the derived quantity and a recapitulation of the *necessary* relation between different kinds of quantity rather than the statement of a relation between units which we find convenient. (Tolman 1917, 249)

The dimensional relations between quantities are *necessary*, not conventional. This distinguishes *quantity* dimensions from unit dimensions, or dimensional systems from unit systems (see §1.3). Generally speaking a dimensional system or a unit system can be used to fix the other, bey defining a coherent system of units. Non-standard dimensional systems are often defined in this way by setting a constant equal to one and eliminating one kind of unit for another, e.g. the spatialization of time *units* in relativity theory upon the adoption of the light postulate, if one takes this to be a true *elimination* of the constant c then one adopts a *dimensional* system in time and length units are equivalent.[56] Tolman rejects any such conventionalism regarding the basic quantity dimensions. For him the reduction of the time dimension to space dimensions would be the same as reducing pressure to volume on account using them to form a two dimensional graph—a well founded correlation is insufficient for a dimensional reduction, let alone the reduction of a *basic* quantity dimension.[57] By distinguishing the necessary dimensional relations of quantities from the conventional "dimensional" relations of units, Tolman takes himself to be reiterating what I am calling the ontic-representational dimension distinction he made in Tolman (1916). This confusion between the "dimensions of quantity" and "dimensions of unit" he claims may be "a contributory cause for a number of criticisms which have been made on the principle of similitude." (Tolman 1917, 251) That said, Tolman stops short of an explicit defense of his principle, and as far as I've seen, never defends or makes use of it again—at this point his work in GR would be all the more pressing, setting aside the coming quantum revolution. As I will argue in the next section, the points he makes against Bridgman's libertine conventionalism does point the way to a metaphysics of quantity dimensions, but one weaker than the quantity dimension fundamentalism that he develops over the course the debate concerning his principle of similitude.

2.5 Verdicts

As mentioned above, the failure of Tolman's principle of similitude was overdetermined. There is, however, much to learn about the foundations of dimensional analysis from the debate concerning its relation to the principle of similitude and the objectivity of ontic quantity dimensions. Here are the results we may take from each of the criticisms discussed above.

[56] Physicists often talk in this manner, but it is apparent that they usually take this to only be a change in unit systems and not in dimensional systems. The "suppressed" constants return when it is time for physical interpretation (compare Rücker 1888).

[57] Though Tolman is a metaphysical realist about dimension, he thinks what we take to be the number of dimensions is a manner of empirical inquiry. The special sciences, following the example of thermodynamics, may introduce new kinds of measurable quantities (e.g. economics). The reduction of the number of dimensions seemed to him impossible, but not logically so.

Buckingham correctly shows that the principle of dimensional homogeneity and the Π-theorem which follows from it can generate a broad class of symmetry transformations of which Tolman's "relativity of size" is only a special case (generated by a scale transformation of length and the fixing of the speed of light and Coulomb's law, etc.). Tolman is right to claim that the principle of similitude is the broader principle in different, methodological sense, that it can be used to derive functional equations for systems in which the dimension of the relevant constant is unknown—a situation in which the principle of dimensional homogeneity alone is useless.

Ehrenfest-Afanassjewa sharpens the criticism that Tolman's principle is merely setting up different dimensional system from the standard one embodied in the C.G.S. unit system. She argues that while such a system may be set up, Tolman has not met the conditions needed to give what is in the first instances a *unit* transformation an *ontic* interpretation—one such condition will be a change in the magnitude of the gravitational constant across the similitude transformation, a transformation she takes to be nomologically impossible. Tolman capitulates that his principle only works as setting up a representational system of units—though this may still constrain the form of future theories of gravity—and puts forward a robust, realist (what I call fundamentalist) metaphysics of dimensions.

Bridgman shows this extra domain of applicability to *not* be an argument in favor of the methodological priority of Tolman's principle of similitude, contrary to Tolman's rejoinder to Buckingham. For one, the epistemic benefit of the principle is limited as it depends on an assumption about the dimensions of the relevant constant, though not its exact dimensional formula: its dimensions must be such that it is invariant under the similitude transformation. While this turns out to generally be the case (with the notable exception of G), Bridgman shows that given the number of constants and the number of basic dimensions ("fundamental units") any principle of similitude based on the scaling of a single such basic dimension would lead to *some* constant or another being left out, depending on which laws are held to be invariant under the transformation. The similitude transformations follow from this conventional choice and dimensional homogeneity (or more directly, unit invariance), and Tolman's chosen unit system fails to be empirically adequate in the case of gravity. Tolman, systematizing his response to Ehrenfest-Afanassjewa, does not fully defend the principle of similitude but aims to clarify a confusion that he takes to be behind criticisms of the principle levied by Bridgman and others. Tolman distinguishes between ontic quantity dimensions and formal unit dimensions and claims that which Bridgman's conventionalist argument depends on a confusion between the two. While unit systems are indeed conventional, dimensional systems, expressed by dimensional formulae, are supposed to be representative of the intrinsic metaphysical nature of the quantities they describe: we cannot conventionally chose the basic quantity dimensions. Insofar as Tolman retreated to a unit dimension interpretation of his principle in response to Ehrenfest-Afanassjewa's criticism, this marks a complete rejection of the ontic interpretation of the principle of similitude, but it also marks the beginning on a debate regarding the metaphysics of quantity dimensions that has largely been neglected.

3 Three Metaphysics of Quantity Dimensions

In this section I summarize the two metaphysical accounts of quantity dimensions which emerge from the early methodological debate and propose a synthesis which overcomes difficulties with both positions. As described in §1.4, fundamentalism, the metaphysics of dimensions espoused by Tolman, and conventionalism, the anti-metaphysics espoused by Bridgman, can be understood as opposite positions regarding two theses:

(1) Quantity dimensions are substantive natural kinds.

(2) The basis of quantity dimension space is determined by nature.

The fundamentalist accepts both theses, and the conventionalist rejects both theses. The conventionalist case against (1) lies in the fact that the use of dimensions to reason about unit transformations is sufficient to explain their usefulness and sufficient to capture all of the results of dimensional analysis. The conventionalist case against (2) relies on the symmetry in defining equations: we can just as well take $f = ma$ to define the force dimension in terms of the dimensions of mass and acceleration as we can take it to define the mass dimension in terms of the dimensions of force and acceleration. Therefore the definition of a quantity dimensions does not say anything about its "essential nature", it pure convention which dimensions are treated as basic in the first place. Further, the conventionalist argues that we can make either reductive definitions of dimensions that eliminate basic quantity dimensions or add basic quantity dimensions from which new derivative quantity dimensions can be defined: the number of basic quantity dimensions is also determined by convention.

In order to make clearer the fundamentalist rejoinder, I divide (2) into two sub theses, yielding three fundamentalist commitments:

(1) Quantity dimensions are substantive natural kinds.

(2a) The number of and the identity of the basic quantity dimensions are determined by nature.

(2b) Relations between different quantity dimensions, as in the defining equations of derivative dimensions, are necessary and not conventional.

The conventionalist argument against (2a) is only partially successful—while there appears to be no natural constraint on *which* quantity dimensions appear as fundamental. A dimensional system for mechanics which treats force as a basic quantity *is as representationally adequate as* a dimensional system which treats mass as a basic quantity instead. However, there is a natural lower limit on the number of quantity dimensions that can adequately represent a physical system. In fact, in Tolman's rebuttal to Bridgman's conventionalism, he puts forward the essential argument in favor of the objectivity of basic quantity dimensions: the Rayleigh-Riabouchinsky paradox. The paradox shows us that artificially reducing quantity dimensions reduces the power of the principle of dimensional homogeneity. For example, Tolman (1917, 250) shows that the dimensional analytic derivation of the equation for the centripetal force,

$$f = k\frac{mv^2}{r},$$

becomes much more indeterminate when the dimensions of length and time are equated (reducing the basic mechanical dimensions to two by making velocity (c) dimensionless):

$$f = k\frac{mv^n}{r}.$$

This is evidence that a dimensional system which collapses the length and time dimensions lacks the representational capacity to describe the centripetal force—this is a constraint set by *nature*. However, Tolman went to far in holding that this shows that the the identities of the basic quantity dimensions are objectively determined by nature; it is in fact the *number* of basic dimensions that are so determined.

3.1 The Generalized Rayleigh-Riabouchinsky Paradox: The Nature of Dimensional Explanations

In an early exposé of dimensional analysis, Rayleigh (1915) uses dimensional analysis to derive equations for a number of systems including a case of heat transfer between a rigid rod and a stream of fluid. Riabouchinsky (1915) showed that by reducing the number of dimensions involved in describing the system from four to three—by eliminating the independent dimension of temperature due to an adoption of the mechanical theory of heat—dimensional analysis results in a less determinant result. It would seem then that we have a paradox: *more* knowledge about the system, that temperature has equivalent dimension to energy, yields a *less* informative result![58] This surprising result shows that not all dimensional formulae can be understood as reductive, and so the multiplicity of a dimensional system is not fully conventional but rather is restricted on one side by nature—the elimination of an independent, basic temperature dimension leads to an inadequate representation of the heat transfer system.

The consider a purely mechanical (and simpler) variant of the Rayleigh-Riabouchinsky paradox from Bridgman's *Dimensional Analysis*.[59] The system that we wish to describe is an *elastic* pendulum (we are only dealing with vertical motion here so this is wholly distinct from the simple pendulum discussed in §1.1): a cubic box of volume v, filled with liquid of density d, is hung from a ceiling by

[58]See Palacios (1964) and Gibbings (2011) for treatments of the paradox in light of a fuller understanding of the algebra of dimensions. "Many authors believe to have made an important discovery in forming dimensional systems with the aid of an insufficient basis. To think so is akin to maintaining that the details of a scene would be better appreciated from its shadow projected on a screen, rather than by looking at it directly." Palacios (1964), 43.

[59]The relevant discussion begins on Bridgman (1931), 59. I follow it closely.

a spring with an elastic constant k. The mass of the liquid in the box is acted on by gravity with a constant acceleration of g. We are tasked with determining the period of oscillation of the pendulum, t. First we write down the dimensional formulae for the variables that describe the system:

$$\begin{aligned}[k] &= \mathrm{MT}^{-2} \\ [t] &= \mathrm{T} \\ [v] &= \mathrm{L}^3 \\ [d] &= \mathrm{ML}^{-3} \\ [g] &= \mathrm{LT}^{-2}.\end{aligned}$$

Since this system is described by five variables and the mechanical dimensional system involves three basic dimensions, dimensional analysis will determine the form of two dimensionless products of powers of the dimensional quantities involved (i.e. Π-terms). To determine the forms of the Π-terms, we must solve two sets of equations for the dimensional exponents of the component terms. Each set is composed of equations for each basic dimension. Three equations and five variables means that the exponents of two variables must be arbitrarily determined. We follow Bridgman in choosing the simplest cases for each Π-term: $\Pi_1 \propto t^1 k^0$ and $\Pi_2 \propto t^0 k^1$. So the Π-terms will each have the form:

$$\begin{aligned}\Pi_1 &= t v^{\alpha_1} d^{\beta_1} g^{\gamma_1} \\ \Pi_2 &= k v^{\alpha_2} d^{\beta_2} g^{\gamma_2}.\end{aligned}$$

Now we set up the two sets of linear equations to determine exponents of zero in each basic dimension for the two Π-terms:

$$\begin{aligned}\mathrm{M}: &\quad 0\alpha_1 + \beta_1 + 0\gamma_1 + 0 = 0 &\qquad 0\alpha_2 + \beta_2 + 0\gamma_2 + 1 = 0 \\ \mathrm{L}: &\quad 2\alpha_1 - 3\beta_1 + \gamma_1 + 0 = 0 &\qquad 3\alpha_2 - 3\beta_2 + \gamma_2 + 0 = 0 \\ \mathrm{T}: &\quad 0\alpha_1 + 0\beta_1 - 2\gamma_1 + 1 = 0 &\qquad 0\alpha_2 + 0\beta_2 - \gamma_2 - 2 = 0.\end{aligned}$$

These equations yield these two solutions,

$$\begin{aligned}\alpha_1 &= -\frac{1}{6} &\qquad \alpha_2 &= -\frac{2}{3} \\ \beta_1 &= 0 &\qquad \beta_2 &= -1 \\ \gamma_1 &= \frac{1}{2} &\qquad \gamma_2 &= -1,\end{aligned}$$

so

$$\begin{aligned}\Pi_1 &= tv^{-\frac{1}{6}}g^{\frac{1}{2}} \\ \Pi_2 &= kv^{-\frac{2}{3}}d^{-1}g^{-1}.\end{aligned}$$

The Π-theorem tells us that the system can be described by an equation of the form

$$0 = \Psi(\Pi_1, \Pi_2),$$

where Ψ is an undetermined function. This equation can be restated[60] as a solution for Π_1

$$\Pi_1 = f(\Pi_2),$$

which yields an equation for t:

$$t = v^{\frac{1}{6}}g^{-\frac{1}{2}}f\left(\frac{kv^{-\frac{2}{3}}}{dg}\right),$$

where f is some undetermined function.

Rather than define volume as a derivative dimension in terms of length we can treat it as an independent dimension (with its own fundamental unit) with a perhaps surprising result.

$$\begin{aligned}[k] &= \mathrm{MT}^{-2} \\ [t] &= \mathrm{T} \\ [v] &= \mathrm{V} \\ [d] &= \mathrm{MV}^{-1} \\ [g] &= \mathrm{LT}^{-2}\end{aligned}$$

Five variables and four basic dimensions yields a single Π-term of the form:

$$\Pi = tk^{\alpha}v^{\beta}d^{\gamma}g^{\delta}.$$

We solve the set of linear equations,

$$\begin{aligned}\mathrm{M}: &\quad \alpha + \gamma = 0 \\ \mathrm{L}: &\quad \delta = 0 \\ \mathrm{T}: &\quad -2\alpha - 2\delta + 1 = 0 \\ \mathrm{V}: &\quad \beta - \gamma = 0\end{aligned} ,$$

[60]See Buckingham (1914), 351 and Bridgman (1931), 41.

yielding the solutions $\alpha = 1/2$, $\beta = -1/2$, $\gamma = -1/2$, and $\delta = 0$. So

$$\Pi = tk^{\frac{1}{2}}v^{-\frac{1}{2}}d^{-\frac{1}{2}}$$

and solving for t:

$$t = C\sqrt{\frac{vd}{k}},$$

where C s a dimensionless constant. This solution is consistent with *but more determinate than* the equation derived when we include the fact that volume is reducible to length *a priori* (as an axiom of geometry). Bridgman notes that rather than introduce volume as a basic dimension, we could have used the fact that the volume and the density of the liquid matters for the behavior of the system only insofar as they determine the mass of the filled box. With three dimensions and only four variables, we would have derived the equally determinate $t = C\sqrt{\frac{m}{k}}$.

The framework of dimensional analysis provides a ready explanation of this phenomenon: By decreasing the number of basic dimensions but leaving the number of variables the same, Riabouchinsky raised the number of Π-terms needed to describe the heat transfer system; By increasing the number of basic dimensions relative to the number of variables, Bridgman reduced the number of Π-terms needed to describe elastic spring system. Practicalities aside, one may question the warrant of such moves—which the conventionalist allows as a matter of convenience and convention. As clarified in Bridgman (1931): basic dimensions may be increased by adopting further dimensional constants and basic dimensions may be reduced by the adoption of non-empirical laws defining one quantity dimension in terms of others, thereby eliminating a dimensional constant.[61] Whether such manipulations are so innocent is put into question by the difference in the power of dimensional analysis depending on how the dimensional system is tuned.

Recent philosophers of science have provided accounts of how dimensional analysis provides explanations, and in doing so have attempted to eliminate any sense of "paradox" from the Rayleigh-Riabouchinsky phenomena discussed above. Lange (2009) has argued that dimensional analysis provides explanations of derived laws which screen off the fundamental laws. Dimensional analysis explains certain similarity features of systems that are independent of various aspects of their constitution (and so the sometimes distinct sets of fundamental laws that govern the phenomena in question).[62] I want to emphasize something about how his account of dimensional explanations

[61]"Temperature may be either chosen as an independent unit, when the gas constant will appear explicitly as a variable, or temperature may be so defined that the gas constant is always unity, and temperature has the dimensions of energy. The same procedure is not incorrect in problems not involving the gas constant in the solution. But if in this class of problem temperature is defined as equal to the kinetic energy of an atom (or more generally equal to the energy of a degree of freedom) and the gas constant is made equal to unity, the fundamental units are restricted with no compensating advantage, so that although the results are correct as far as temperature is proportional to the energy of a degree of freedom, they will not give so much information as might have been obtained by leaving the units less restricted." (Bridgman 1931, 72)

[62]Lange considers the dimensional similarities of waves in a fluid and standing waves in a string (Lange 2009, sec. 4.) Lange also discusses the role of dimensional explanations with respect to meta-laws or symmetry principles.

applies to the generalized Rayleigh-Riabouchinsky paradox. Derivations which include a higher ratio of basic dimensions to variables apply to a larger set of cases; they apply to systems independently of the value of the dimensional constants that link the various basic dimensions together in the laws. In the Bridgman case, treating volume as a basic dimension independent of length allows for the derived equation to apply even in non-Euclidean geometries where $v = l^3$ may not hold.[63] As an empirical matter, the would-be dimensional volume constant ω, where $v = \omega l^3$ and $[\omega] = \mathrm{VL}^{-3}$, could have a value other than 1. In the thermodynamic case, it could[64] be that the value of Boltzmann's constant or the gas constant was different, such that a unit of temperature would not be equivalent to a unit of energy. In both cases, the derivation that allows for the possibility of the variation of these constant, i.e. does not treat the relevant laws as *a priori*, is the more explanatorily powerful as it is more general.[65] Pexton (2014) gives a different, though consistent account of how dimensional analysis explains: dimensional analysis provides *models* of systems that make apparent patterns of *modal* dependence (i.e. counterfactual) on the quantities. On Pexton's modal model theory of dimensional explanations, Rayleigh-Riabouchinsky phenomena can be accounted for by the fact that for some systems such dimensional reductions, like that of temperature to energy, are simply *irrelevant*. It is no surprise that irrelevant factors can introduce noise (in this case in the form of extra degrees of freedom) that interfere with the power of an explanation given by the model. As seen with Lange's account, there is a tradeoff between abstraction and explanatory power.

The conventionalist makes both the general explanatory power and also the differences in explanatory power depending on dimensional system (as revealed by the Rayleigh-Riabouchinsky paradox) mysterious. Surely if some choice of convention is better than another, not as a matter of what is convenient to deal with, but in its *representational capacities*, we ought question whether dimensional systems are indeed a matter of convention after all. The fundamentalist has a nicer story to tell about the explanatory and representational power of dimensional analysis: dimensions exist and some dimensional systems better describe their natures than others. However, the conventionalist critique still has some bite. Generally, the Rayleigh-Riabouchinsky paradox only shows that the *number* of basic quantity dimensions, the degrees of freedom in the dimensional system, is constrained by nature. Both practice and mathematical theory (see §3.3) give reason to believe that the basis of a dimensional system is not unique. This conventionalist constraint on our metaphysics of quantity dimensions can be seen by considering the symmetric nature of defining equations: the relation between volume and length is equally well expressed by the formulae $\mathrm{V} = \mathrm{L}^3$ and $\mathrm{L} = \mathrm{V}^{\frac{1}{3}}$. What is

[63] This fails to hold in a very mundane case: A liter of volume was defined (by the CGPM in 1901, until 1964) as the volume occupied by a kilogram of pure water in standard conditions, rather than as a cubic decimeter, as it is currently. While the two definitions aim to define the same quantity, the correspondence is not exact, meaning the former definition requires a constant to relate the volume and length unit, and the conceptual independence of volume from length in this this defintiion requires that this constant be *dimensional*. See Petley (1983), 137.

[64] Lange (2009) holds that this is a counterlegal. This depends on the somewhat controversial though underappreciated thesis that the values of the constants are part of the laws, e.g. nomologically necessary. See Jacobs (Forthcoming) and Jalloh (Forthcoming) for reasons why this may not be the case.

[65] The explanation is powerful because it applies to more possible (or impossible) worlds; the derivation has greater modal strength.

needed is a metaphysics of dimensions that captures the objective *structure* of dimensional systems while leaving open for convention a choice of basis. Further, this structure needs to be such that it provides a foundation for the representational and explanatory success of dimensional analysis. In the next section I introduce such a metaphysics of quantity, a moderate realism, quantity dimension functionalism.

3.2 Functionalism: Best of Both Worlds?

To sum up the state of play: Even in the face of conventionalism we can accept thesis (2b) without restraint. The realist must, however, attenuate (1) and (2a). From (1) we can salvage the ontological realism of the quantity dimensions, but we must refrain from giving the quantity dimensions intrinsic natures; Quantity dimensions are rather structural features of reality.[66] From (2a) we can salvage that the *number* of basic quantity dimensions is determined by nature, otherwise our conventionally chosen dimensional system will fail to be empirically adequate or will be redundant and interfere with scientific practice. We again refrain from giving the basic quantity dimensions *identities* or *essences* that are independent of the invariant relations between them as described by the laws. This is an ontic functionalist view regarding quantity dimensions.

While I cannot here hazard a full exposition of a functionalist account of quantity dimensions, a sufficiently precise understanding of the proposal can be had by a consideration of the vector space representation of quantity dimensions. That there is an analogy, indeed an algebraic isomorphism, between quantity dimensions and vectors in a linear space. As such, we can use a vector space to *model* quantity dimensions and their relations—the invariants of this representation guide the functionalist understanding of the nature of quantity dimensions; the symmetries of this representation provide a guide to quantity symmetries.[67] I here follow the development of the vector space representation most clearly made by Corrsin (1951), Palacios (1964), and Johnson (2018).[68]

3.3 Changing the Basis of Mechanical Dimension Space

Rather than give an abstract presentation of the vector space representation of quantity dimensions, I present a vector space corresponding to a familiar dimensional system, the mechanical dimensions.

[66] In case this is not clear: if we take the nature of derivative dimensions to be determined by their relations to to their defining basic dimensions, the conventionality of which dimensions are basic and which are derivative makes the natures of erstwhile derivative dimensions conventional. This can be avoided if the nature of dimensions are not constituted by constructions of the intrinsic natures of the basic dimensions.

[67] For general discussions of quantity symmetries see Roberts (2016) and Jalloh (Forthcoming).

[68] As Palacios and Johnson point out, this representation has its roots in the approach to dimensional analysis initiated by Ehrenfest-Afanassjewa (1926)—Her method of generalized homogeneous functions seems to have first been further developed by San Juan (1947). There is also a somewhat parallel literature in mathematics that reproduces and extends some of the results discussed here: e.g. Whitney (1968a); Whitney (1968b); Tao (2012); Raposo (2018); Raposo (2019). In particular, I will note that a given dimension space is more appropriately represented by a finitely generated Abelian group, which roughly is a vector space with an unchosen basis, see Raposo (2018) for much more. A more comprehensive consideration of mathematical models of quantity dimensions will have to be postponed.

The mechanical dimensions are those dimensions which are reducible to products of powers of basic dimensions mass (M), length (L), and time (T). By representing the basic dimensions as three orthogonal basis vectors, we can represent all mechanical dimensions as vectors in the space spanned by $\vec{\mathrm{M}}$, $\vec{\mathrm{L}}$, and $\vec{\mathrm{T}}$.

We define the "pure" dimensions by unit vectors:

$$\vec{\mathrm{M}} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \vec{\mathrm{L}} = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad \vec{\mathrm{T}} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Derived dimensions can be represented by vectors that are linear combinations of the basis vectors, with the number in each coordinate location being the power of that quantity in the corresponding basic dimension. Some examples of derived quantity dimension vectors:

$$\vec{\mathrm{V}} = \begin{pmatrix}0\\1\\-1\end{pmatrix}; \quad \vec{\mathrm{P}} = \begin{pmatrix}1\\2\\-2\end{pmatrix}; \quad \vec{\mathrm{F}} = \begin{pmatrix}1\\1\\-2\end{pmatrix},$$

where $\vec{\mathrm{V}}$ is the velocity dimension vector, $\vec{\mathrm{P}}$ is the power dimension vector, and $\vec{\mathrm{F}}$ is the force dimension vector.[69]

Now some of my earlier remarks regarding a functionalist theory of quantity dimensions can be made more clear. First, taking the mechanical dimension space as our base, we can understand the introduction of basic quantity dimensions (e.g. temperature) as raising the dimensionality of the space. On the other hand a reduction of the space (e.g. a reductionist mechanical theory of heat or spatializing time via c) corresponds to a projection onto a subspace of lower dimensionality. Parsimony and practical use tells against the proliferation of "superabundant" bases. The strict lower limit on the multiplicity of a basis for a dimension space depends on the laws and the number of quantities (derived or basic) needed to describe the system. The use of an insufficient basis leads to a loss of information and representational capacity for the dimensional system.

What is conventional are the identities of the base vectors. This conventionality is familiar from linear algebra. Consider a system in which force replaces mass as a basis vector. The new basis:

$$\vec{\mathrm{F}} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \vec{\mathrm{L}} = \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad \vec{\mathrm{T}} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

[69] See Corrsin (1951) for a graphical representation of mechanical space, also reproduced in Johnson (2018).

This produces new dimensional formulae for (some of) the derived quantities:

$$\vec{\mathrm{V}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; \quad \vec{\mathrm{P}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad \vec{\mathrm{M}} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Now the question is what is the invariance under such a transformation of bases that is supposed to be metaphysically significant? Note the invariance in the relationships between force and mass: regardless of which is treated as basic, the other scales with it by a power of 1. This means that any scale transformation, active or passive, on the mass dimension will propagate to the force quantities as well: if the masses double so will the forces, etc. This change of basis considered only transformed the mass vector among the prior base vectors and so only transforms the dimensional power of quantity vectors composed by the mass: so the coordinates of $\vec{\mathrm{P}}$ are affected while those of $\vec{\mathrm{V}}$ are not. The components of each vector will vary over the change, but the *intrinsic* geometrical properties of the vectors will remain invariant. One such intrinsic property of the vector space is the inner product. The invariance of the inner product to a change of basis means that the magnitudes and the relative angles[70] of the elements of the vector space are invariant, hence: the dependence relations between different quantity dimensions remain invariant. As these dependence relations will be symmetric, some quantity dimensions cannot be said to *ground* others, except relative to a basis, and quantity dimension symmetries[71] will be tightly constrained as they will involve the transformations of all the quantity dimensions with relevant dependency relations. Such quantity dimensions symmetries define a class of dynamical symmetries—dimensional analysis is used to determine *similarity* relations, transformations under which two systems can be used as (dynamic) models of each other (see Sterrett 2009; Sterrett 2017 for details). These dependence relations therefore play a double role of identifying the quantity dimensions *relative to each other* and of constraining the forms of the laws. That they play the first role is determined by linear algebra; That they play the second role was already established in the discussion of dimensional analysis above and is discussed in more detail elsewhere.[72]

[70] One enticing idea that I can explore fully here is this: that the physical constants correspond to the relative angles of quantity dimension vectors. If some fundamental constant, corresponding to the angle between some fundamental dimensions, is set to 1, then two quantity dimensions collapse and there is a reduction of the order of the dimension space, as in "natural units". This is suggested by the treatment given by Raposo (2018), examples 3.8, 4.4, of such transformations.

[71] I.e. active dimensions scale symmetries. See Martens (2021) and Jalloh (Forthcoming) for discussion.

[72] One might quibble here with my "constraining" language. With Campbell (1924) and Palacios (1964), one may argue that the laws constrain dimensional analysis by defining the relations between dimensional quantities. I here do not want to establish any sort of priority claim regarding the structure of the physical dimensions or the forms of the laws; they are mutually constraining and I will only claim one takes precedence over the other depending on the epistemic context.

4 Conclusion

This paper has exposited an unduly neglected debate regarding the methodological and metaphysical foundations of dimensional analysis and has evaluated the merits of the two major positions, conventionalism and fundamentalism. Both positions are found lacking: conventionalism regarding quantity dimensions fails to account for the success of dimensional analysis and the representational constraints on dimensional systems; fundamentalism fails to fit with the conventionality found in scientific practice and fails to give reason to privilege any basis over others for a dimensional system. I've set forth the basic outline of a functionalist account of quantity dimensions, wherein the empirical constraints on the number of basic quantity dimensions and the conventionality regarding *which* quantity dimensions are treated as basic are respected. The presentation of this position is aided by the vector space representation of dimensional systems, which makes clear the isomorphism of different dimensional bases and makes clear the information loss associated with insufficient bases (as realized in the Rayleigh-Riabouchinsky paradox). The metaphysical residue that the functionalist is realist about are the symmetric dependency relations between quantity dimensions, which correspond to the dimensional forms of the laws and so encode *metaphysically robust* scaling relations. The account of functionalism given here is incomplete, my current description of the invariants of a dimensional space is not fully satisfying, and a complete story must say something about the constants, the number of which is correlated with the order of the dimensional system.[73] While the nature of the constants and their relation with the laws has been given some limited attention[74] a full philosophical account will do well to consider the vector representation of quantity dimensions presented here and its relation to the nomological structure of the physical world.

[73] It is well noted in the literature that the addition or elimination of a basic quantity dimension requires the introduction or elimination of a dimensional constant in order to make coherent laws involving that quantity and the other quantities in the system (see Bridgman 1931; Johnson 2018; Gibbings 2011).

[74] See Dahan (2020); Johnson (2018); Jalloh (Forthcoming).

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